CHAPTER III *THEORY OF ELASTICITY*

3.1 Simple laws and linear elasticity

Consider a rectangular bar comprising a homogeneous and isotropic material of initial length *l* in the direction of x and dimensions *h* and *w* in y and z cartesian coordinates, respectively. The bar is fixed at the top end (origin), and the force F at the free bottom end (point A) pulls on it in the x-direction (see Figure 3-1).

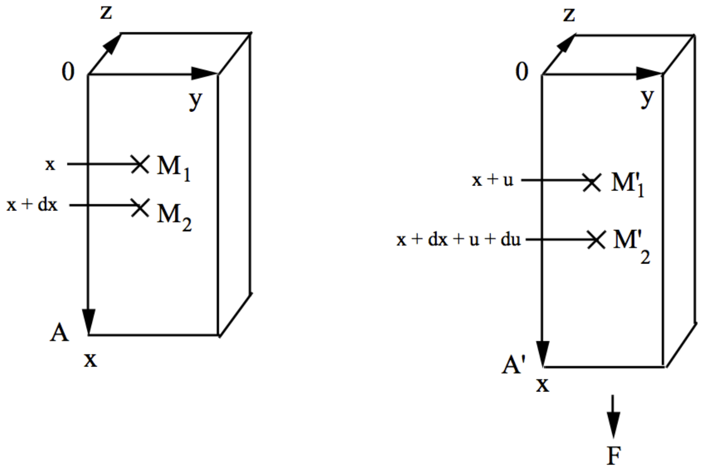


Figure 1-3: *Deformation of a bar*

The application of the force F moves point $\boldsymbol{M_1}$, initially at coordinate $\boldsymbol{x}$, to $\boldsymbol{M'_1}$ at $\boldsymbol{x+u}$. The point $\boldsymbol{M_2}$, close to M_1 at $\boldsymbol{x+dx}$, moves to $\boldsymbol{M'_2}$ at $\boldsymbol{x+dx+u+du}$. The strain of the segment $\boldsymbol{M_1}$-$\boldsymbol{M_2}$ is defined as the ratio:

$$\varepsilon = \frac{M_1'M_2' - M_1M_2}{M_1M_2} = \frac{du}{dx} \tag{3.1}$$

When the strain occurs within the elastic domain, it follows ***Hooke's law*** (1678), which describes a proportional relation between stress ($\boldsymbol{\sigma}$) and strain (ε),

$$\sigma = (E\varepsilon) \tag{3.2}$$

where $\boldsymbol{\sigma = F/S}$, $\boldsymbol{S}$ is the cross-sectional area of the bar, and $\boldsymbol{E}$ is the proportionality constant, called "***Young modulus***" (1807) or elastic modulus.

In this configuration, uniaxial deformation, the section of the bar is constant in the x-direction, and thus the stress σ is also constant. If the total deformation of the bar is $\boldsymbol{\Delta l}$, we have:

$$\boldsymbol{\varepsilon = \Delta l / l}$$

The deformation, Δl in the x-direction, causes contraction of the bar's other sides, $\boldsymbol{\Delta h}$ and $\boldsymbol{\Delta w}$. If the solid is isotropic, these changes in the lateral dimensions are also proportional to the stress (σ),

$$\Delta w / w = \Delta h / h = -\nu \Delta l / l = -\nu \sigma / E \tag{3.3}$$

where ν is a proportionality constant called the "***Poisson ratio***" (first described in 1814), this constant expresses the fact that atoms tend to keep a relative distance that varies only slightly during the deformation in the elastic domain, so the volumetric changes are small.

Because the governing relations are linear for forces and displacements, the ***superposition principle*** holds: the strain produced by two forces, F_1 and F_2, equals the sum of the strains that each force would generate independently. ***This property forms the foundation of linear elasticity; no further assumptions are required.*** For a homogeneous and isotropic material, the elastic response is fully characterized by only two constants: the ***Young's modulus*** (***E***) and the ***Poisson's ratio*** (**ν**).

Applications:

a) hydrostatic pressure

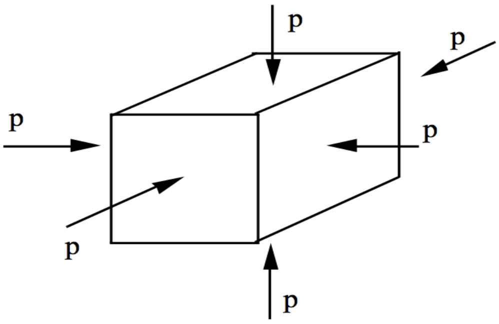


Figure 3-2: *Hydrostatic pressure*

In this case, the pressure (p) is the same on every external surface as if the bar is immersed in a liquid. The variations, Δl, Δh, and Δw, are obtained by considering the stresses separately in the x, y, and z directions and then adding the effects together, that is:

$$\frac{\Delta h}{h} = -\frac{p}{E} + \nu\frac{p}{E} + \nu\frac{p}{E} = -\frac{p}{E}(1-2\nu) \tag{3.4}$$

$$\Delta h / h = \Delta w / w = \Delta l / l$$

$$\Delta V / V = \Delta h / h + \Delta w / w + \Delta l / l = -3(1-2\nu) p / E \tag{3.5}$$

The ***bulk modulus*** K is defined by: $$p = -K\frac{\Delta V}{V} \tag{3.6}$$

$$K = \frac{E}{3(1-2\nu)} \tag{3.7}$$

With this relation, it is clear that for $\nu < 0.5$, $K > 0$. In fact, $K < 0$ would correspond to an expansion of the solid ($\Delta V > 0$) under the action of pressure, which would mean that the solid can provide energy to its surroundings in terms of thermodynamics. For many metals, $\nu \approx 0.33$ (relatively incompressible) and $K \approx E$. Natural rubber is close to ν = 0.5, as well as liquids. We discuss this further in section 3.6.1, where Poisson's constant can range between $-1 < \nu < 0.5$.

b) pure shear

The case of pure shear can be obtained, for example, from the system of forces *F* applied to a cube of side length *L*, as in the Figure below:

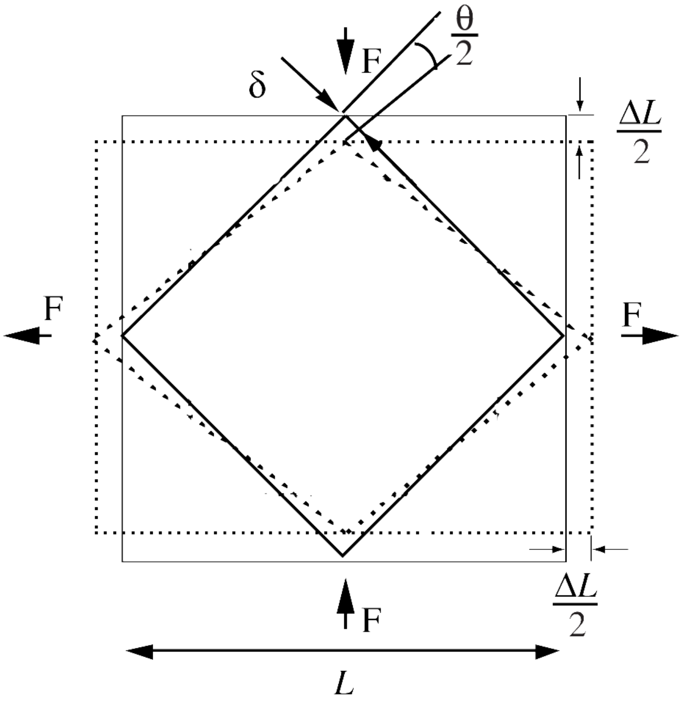


Figure 3-3: *Pure shear plane state*

In this case, we have :

$$\frac{\Delta L}{L} = \pm \frac{F}{S}\left(\frac{1+\nu}{E}\right) \tag{3.8}$$

This configuration of applied forces produces a shear stress of σ = *F/*S on the planes of the cube at a 45 ° angle. As a result, the faces of the cube are turned by an angle of θ/2 so that:

$$\tan\frac{\theta}{2} \approx \frac{\theta}{2} = \frac{4\delta}{L\sqrt{2}} = \frac{\Delta L}{L} \tag{3.9}$$

$$\theta = 2\sigma\left(\frac{1+\nu}{E}\right) \tag{3.10}$$

We characterize this by a shear modulus $\boldsymbol{\sigma=\mu\theta}$, so that:

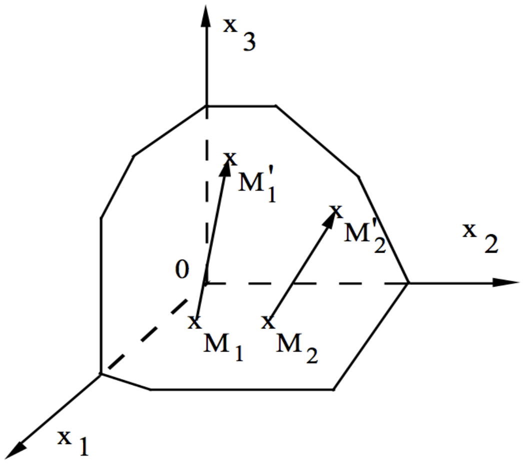
$$\mu = \frac{E}{2(1+\nu)} \tag{3.11}$$

and $\boldsymbol{\mu = 3/8E}$ for $\boldsymbol{\nu = 1/3}$ (ideal isotropic material).

3.2 Strain tensor

In most cases, deformation is a combination of both plastic and elastic deformation. Therefore, it is appropriate first to provide a general definition of the strain tensor that can be applied to both plastic and elastic deformations, and then to give specific definitions and conditions for each case. This strain tensor approach is discussed in detail in the following.

a) Strain tensor definition



Deformation is the response of a material to stress loads. We consider two neighboring points, M_1 and M_2, in a generic solid (Figure 3-4) so that $\overrightarrow{M_1M_2} = \overrightarrow{dx}$. A system of external forces is applied to this solid so that M_1 moves to $M'1$ and M_2 moves to M'_2.

$\overrightarrow{M_1M_1'} = \vec{u}(M_1)$ represents the displacement of point M_1 and, in the same way $\overrightarrow{M_2M_2'} = \vec{u}(M_2)$ represents the displacement of point M_2.

Figure 3-4: *Local deformation of a vector in a solid*

If dx_i [1] represents the vector $\overrightarrow{M_1M_2}$, after deformation, it becomes:

$$dx_i' = dx_i + du_i \tag{3.12}$$

The distance between two points before deformation is:

$$dl = \sqrt{dx_1^2 + dx_2^2 + dx_3^2} \tag{3.13}$$

and after deformation, it is:

$$dl' = \sqrt{dx_1'^2 + dx_2'^2 + dx_3'^2} \tag{3.14}$$

The first-order approximation is:

$$du_i = \frac{\partial u_i}{\partial x_k} dx_k = \beta_{ik} dx_k \tag{3.15}$$

$\boldsymbol{\beta_{ik}}$ is called the displacement gradient tensor. We can also write the second-order approximation:

---

[1] We can use Einstein notation where the components of vectors are given by indexes and where the sum is represented by a repetition of a same index.

$$dl'^2 = dl^2 + 2\frac{\partial u_i}{\partial x_k}dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}dx_k dx_l \tag{3.16}$$

As this expression contains the sum of the indices i,k, and l, it can be simplified as follows:

$$dl'^2 = dl^2 + 2u_{ik}dx_i dx_k \tag{3.17}$$

or

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_k}\frac{\partial u_l}{\partial x_i}\right) \tag{3.18}$$

For small deformations, the term $\frac{\partial u_l}{\partial x_k}\frac{\partial u_l}{\partial x_i}$ can be generally neglected, and thus:

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{3.19}$$

where u_{ik} is called the strain tensor. By its symmetry, it can be diagonalized, and thus, its components along the principal axis ($\overline{x_1}, \overline{x_2}, \overline{x_3}$ coordinates) can be found:

$$dl'^2 = (\delta_{ik} + 2u_{ik})dx_i dx_k = (1+2u^{(1)})d\overline{x}_1^{\,2} + (1+2u^{(2)})d\overline{x}_2^{\,2} + (1+2u^{(3)})d\overline{x}_3^{\,2} \tag{3.20}$$

The comparison between (3.20) and (3.14) gives insight into the physical meaning of this expression. As a matter of fact, by the identification of terms:

$$dx_i' = \sqrt{(1+2u^{(i)})}dx_i \approx (1+u^{(i)})dx_i \tag{3.21}$$

The change in volume of an element $dV = dx_1 dx_2 dx_3$ is then:

$$dV' = dV(1+u^{(1)})(1+u^{(2)})(1+u^{(3)}) \approx dV(1+u^{(1)}+u^{(2)}+u^{(3)}) = dV(1+u_{ll}) \tag{3.22}$$

The trace of the strain tensor (invariant) represents the variation in the volume of a deformed element. The symmetrization of the strain tensor consists of a change of basis. To simplify the concept, consider elastic deformations in a two-dimensional space.

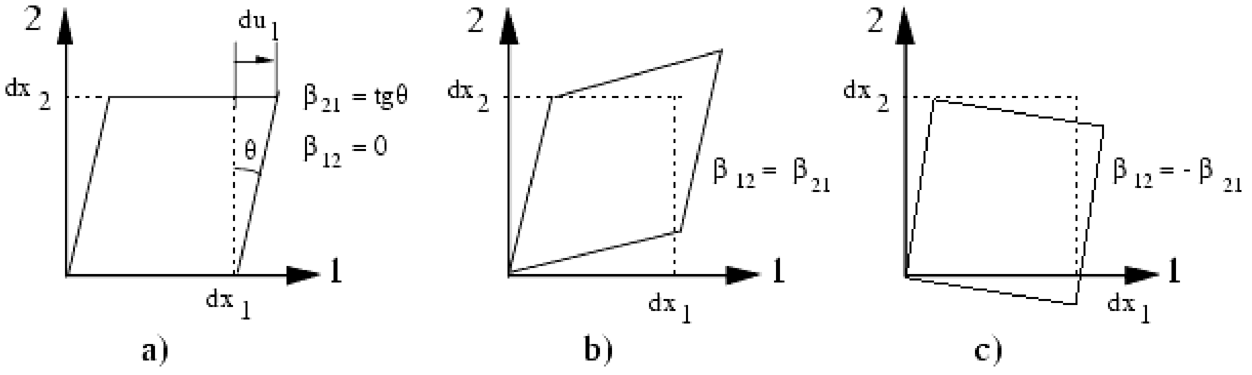


Figure 3-5: *Decomposition of pure shear with a symmetric deformation and a rotation*

We note that the displacement gradient tensor $\boldsymbol{\beta_{ik}}$ is not necessarily symmetric, as in the case of pure shear (Figure 3-5a). We can see that the distortions can be thought of as the sum of deformations (Figure 3-5b) and rotations (Figure 3-5c). It also appears that rotations, which do not contribute to the elastic deformations of the solid, correspond to the antisymmetric[2] part of the tensor $\overline{\boldsymbol{\beta}}$, while deformations correspond to the symmetric part of $\overline{\boldsymbol{\beta}}$ -in reality, this is true only when the deformations are small.

For small deformations, we define the following:

$$\omega_{ik} = \frac{1}{2}\left(\beta_{ik} - \beta_{ki}\right) = -\omega_{ki} \qquad \text{and} \qquad u_{ik} = \frac{1}{2}\left(\beta_{ik} + \beta_{ki}\right) \tag{3.23}$$

where $\boldsymbol{\omega_{ik}}$ are the components of the tensor $\overline{\overline{\boldsymbol{\omega}}} = \overline{\overline{\boldsymbol{\beta}}}^{\boldsymbol{a}}$, which represents the rotation of the solid as a rigid body, and the $\boldsymbol{u_{ik}}$ are the components of the strain tensor $\overline{\overline{\boldsymbol{u}}} = \overline{\overline{\boldsymbol{\beta}}}^{\boldsymbol{s}}$.

3.3 Stress tensor

The resultant of the forces acting on a specific volume of a deformed body is given by:

$$\int \boldsymbol{F}_i dV \tag{3.24}$$

The forces $\boldsymbol{F_i}$ are internal forces in the neighboring region next to the volume being considered. We assume that the force can be transmitted only on the surface of this volume. Thus, the divergence theorem must be applicable:

$$\int \boldsymbol{F}_i dV = \int \frac{\partial \sigma_{ik}}{\partial x_k} dV = \oint \sigma_{ik}\, ds_k \tag{3.25}$$

We call $\boldsymbol{\sigma_{ik}}$ ***stress tensor***. It can be shown that this tensor is symmetric to the strain tensor by calculating the moment of the forces acting on the volume dV. The equilibrium of the solid becomes simply $\boldsymbol{F_i} = \boldsymbol{0}$ and:

$$\boxed{\frac{\partial \sigma_{ik}}{\partial x_k} = 0} \tag{3.26}$$

3.4 Thermodynamics review

The variation of internal energy of a thermodynamic system is given by:

$$dE = \delta Q + \delta W \tag{3.27}$$

where $\boldsymbol{\delta Q}$ is the heat given to the system, and $\boldsymbol{\delta W}$ is the work done on the system. These two quantities are not exact total differentials. In particular, we define $\delta Q \leq TdS$, and the equal sign holds only in a reversible process.

---

[2] A tensor is called antisymmetric if it alternates sign (+/−) when any two indices of the subset are interchanged.

The following considers first those processes and takes into account the classic thermodynamic variables $P, V, S, and\ T$. The internal energy is defined as:

$$dE = TdS - PdV \tag{3.28}$$

If the **volume is constant**, then $dE = dQ$. The internal energy, E, is the most appropriate form of expression for the energy of an adiabatic system with a constant volume.

If the **pressure is constant**, the heat exchanged with the system is given by:

$$dQ = dE + PdV = d(E + PV) = dH \tag{3.29}$$

$$dH = TdS + VdP \tag{3.30}$$

This expression is an exact differential. In an adiabatic process, $dH=0$. H is called enthalpy and represents the most appropriate form of energy for a constant pressure process. The work done while keeping a **constant temperature** in a reversible process is given by:

$$\delta W = dE - dQ = dE - TdS = d(E - TS) = dF \tag{3.31}$$

$$dF = -PdV - SdT \tag{3.32}$$

$\boldsymbol{F = E - TS}$ is the thermodynamic free energy (Helmholtz free energy) and represents the most appropriate form of energy for an isothermal system at constant volume if the process is nonreversible, $\delta Q < TdS$ and thus $F > \delta W$.

Finally, if the **pressure and temperature are constant** (common conditions for a laboratory), we use the Gibbs free energy:

$$G = E - TS + PV = H - TS = F + PV \tag{3.33}$$

$$dG = -SdT + VdP \tag{3.34}$$

The physical meaning of G can be grasped by considering systems of N variable particles. We then introduce the chemical potential μ of a particle, and the internal energy variation becomes:

$$dE = dQ + dW + \mu dN \tag{3.35}$$

As ***T, P,*** and N are independent, it can be rewritten as:

$$dG = -SdT + VdP + \mu dN \tag{3.36}$$

G is the chemical potential of the system.

$$G = \mu N \tag{3.37}$$

How are thermodynamic functions used?

In a nonreversible process $\frac{\delta Q}{dt} \leq T\frac{dS}{dt}$, entropy increases without adding heat.

$$\frac{\delta Q}{dt} = \frac{PdV}{dt} + \frac{dE}{dt} \Rightarrow \frac{PdV}{dt} + \frac{dE}{dt} \leq T\frac{dS}{dt} \tag{3.38}$$

If *T* and *V* are constants, then

$$\frac{d(E-TS)}{dt} = \frac{dF}{dt} \leq 0$$

The system tends to a minimum of free energy, ***F***.

If *P* and *T* are constants

$$\frac{d(E-TS+PV)}{dt} = \frac{dG}{dt} \leq \frac{VdP}{dt} = 0 \tag{3.39}$$

The system tends to a minimum of ***G***.

If *V* and *S* are constants:

$$\frac{dE}{dt} = \frac{\delta Q}{dt} \leq TdS = 0 \quad \Rightarrow \quad dE = 0 \tag{3.40}$$

The internal energy *E* is minimal, and the process is reversible. Therefore, the quantities of *F* and *G* can be used to describe irreversible processes, which will be shown in Chapter 7, and descriptions of plastic deformation from dislocation motion.

3.5 Thermodynamics and Deformation

3.5.1 Work

Consider a deformed body that is left without external stress applied. The deformation $\boldsymbol{u_i}$ varies of a quantity δu_i.

The work δW done by the internal forces $F_i = \frac{\partial \sigma_{ik}}{\partial x_k}$ is given by:

$$\int \delta W dV = \int \frac{\partial \sigma_{ik}}{\partial x_k} \delta u_i dV \tag{3.41}$$

The integration by parts yields:

$$\int \delta W dV = \oint_{\Sigma} \sigma_{ik}\, \delta u_i\, ds_k - \int_{V} \sigma_{ik} \frac{d\delta u_i}{dx_k} dV \tag{3.42}$$

The surface integral represents the surface tension of the body, which is negligible in the elasticity of metals. In our case, the forces applied externally to the deformed body are zero and do not contribute to this term. Using the symmetrization of the strain tensor described in equation (3.19):

$$\int \delta W dV = -\int \sigma_{ik}(\delta u_{ik}) dV \tag{3.43}$$

and consequently

$$\delta W = -\sigma_{ik}\delta u_{ik} \tag{3.44}$$

This expression defines the work per unit volume done by the deformation of the solid. We note that $\boldsymbol{\delta W}$ is the matrix product $-\boldsymbol{\sigma_{ik}}$ and the trace $\boldsymbol{\delta u_{ik}}$.

3.5.2 Energy

The variation of the system's internal energy per unit volume dE_v is given by:

$$dE_v = \partial Q_v - \delta W_v = TdS_v + \sigma_{ik} du_{ik} \tag{3.45}$$

The minus sign (-) before the work term is also present in equation (3.44), where the body does work on the surroundings.

Example: uniform compression

In uniform compression, $\sigma_{ik} = -p\delta_{ik}$.

Using equation (3.45) per unit volume, we get:

$$dE_v = TdS_v - p du_{ll} \text{ and thus}$$

$$dE = TdS - p\frac{dV}{V}V = TdS - pdV$$

We find the classic form of internal energy (eqn. 3.28) again. Thus, the equation of (3.45) generalizes the expression for the internal energy. In the following, we do not specify whether the energy is per unit volume; however, the context is sufficient for clarification. The other forms of energy result from the same procedure.

$$dF = -SdT + \sigma_{ik} du_{ik} \tag{3.46}$$

$$dG = -SdT - u_{ik} d\sigma_{ik} \tag{3.47}$$

$$\sigma_{ik} = \left(\frac{\partial E}{\partial u_{ik}}\right)_S = \left(\frac{\partial F}{\partial u_{ik}}\right)_T \tag{3.48}$$

$$u_{ik} = -\left(\frac{\partial G}{\partial \sigma_{ik}}\right)_T = -\left(\frac{\partial H}{\partial \sigma_{ik}}\right)_S \tag{3.49}$$

3.5.3 Physical origin of elasticity

Consider a longitudinal elastic bar kept in strain by a force f. A retraction force balances this force f_r that opposes f. If we consider a small deformation dl, the variation in free energy $F = E - TS$ is given by (3.46), where f is the absolute value of the retraction force.

$$dF = -SdT - PdV + f_r dl \tag{3.50}$$

$$f_r = \left.\frac{\partial F}{\partial l}\right|_{T,V} = \left.\frac{\partial E}{\partial l}\right|_{T,V} - T\left.\frac{\partial S}{\partial l}\right|_{T,V} \tag{3.51}$$

From this expression, we deduce that the elastic retraction force corresponds to the increase in free energy of the system per unit length of elongation. It can be divided into two terms: one contribution resulting from the opposition of the sample to the increase in internal energy, and another from the entropy associated with heat dissipation. In other words, the deformation modifies the distances between atoms, resulting in increased internal energy (the atom positions are no longer a minimum in the bond potential energy) and dissipating heat. We can separate these two effects by studying the variation of the retraction force (elastic modulus) as a function of the temperature. As the free energy F is an exact total differential of T and l:

$$\frac{\partial}{\partial l}\left(\frac{\partial F}{\partial T}\right)_l = \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial l}\right)_T \tag{3.52}$$

which becomes considering (3.50) and (3.51):

$$-\left.\frac{\partial S}{\partial l}\right|_{T,V} = \left.\frac{\partial f_r}{\partial T}\right|_{l,V} \tag{3.53}$$

By combining (3.53) and (3.51)

$$f_r = \left.\frac{\partial E}{\partial l}\right|_{T,V} + T\left.\frac{\partial f_r}{\partial T}\right|_{l,V} \tag{3.54}$$

The expression in (3.54) of the retraction force enables us to distinguish two physical cases: the ideal crystal and the ideal elastomer (natural rubber). In the first case, the retraction force corresponds only to an internal energy variation and remains constant with temperature. In the second case, the retraction force increases proportionally to the increase in temperature.

As a matter of fact, T>0 and $f_r > 0$ and thus also $\frac{\partial f_r}{\partial T} > 0$.

An ideal elastomer heats up when stretched and cools down when released. The physical origin of this behavior comes from the molecular structure of elastomers. These materials are formed from long polymer chains linked with weakly interacting Van der Waals bonds in the order of one interaction for every 100 monomers (see Chapter I). The internal energy of an elastomer is only affected slightly by elongation. Consequently, elongation leads to increased monomer chain alignment of the chains and, thus, a decrease in entropy. Since $\frac{\partial S}{\partial l} \leq 0, T$ increases with the retraction force f_r shown in equation (3.54).

The retraction force is due to the Brownian motion of the monomers, which increases with temperature. We must note the analogy with gases, which warm up when adiabatically compressed. In analogy to (3.51), we have:

$$P = T\left.\frac{\partial P}{\partial T}\right|_V - \left.\frac{\partial E}{\partial V}\right|_T = T\left.\frac{\partial S}{\partial V}\right|_T - \left.\frac{\partial E}{\partial V}\right|_T$$

Most crystal materials show an elastic behavior of enthalpic type, whereas amorphous polymers behave similarly to rubber. In the order of 100 GPa, the elastic modulus of crystal materials is 100 times larger than that of elastomers.

3.6 Hooke's law

3.6.1 Deformation of isotropic solids

To apply general thermodynamic relations found in § 3.5, we must express the energy as a function of the strain tensor. We can derive this equation by expressing the energy in the power series of $\boldsymbol{u_{ik}}$. Consider a solid body that has been deformed at a constant temperature. The thermodynamic potential to consider is the free energy, F. We suppose that, for $\boldsymbol{u_{ik} = 0}$, there are no stresses, so that $\boldsymbol{\sigma_{ik} = 0}$. As in equation (3.48):

$$\sigma_{ik} = \left(\frac{\partial F}{\partial u_{ik}}\right)_T$$

Thus, the development of F does not include linear terms in $\boldsymbol{u_{ik}}$. In an isotropic solid, the deformation energy has to be independent of the direction of deformation with respect to the coordinate system of the solid. In other words, the energy must be invariant to a coordinate transformation.

Invariants

We noted that $\boldsymbol{u_{ik}}$ is a symmetric matrix, and thus, it can be diagonalized. To find the eigenvalues, the characteristic equation to solve is written as:

$$\det(\bar{\bar{u}} - \lambda I) = 0 \Rightarrow \lambda^3 - Tr(\bar{\bar{u}})\lambda^2 - C_2\lambda + \det(\bar{\bar{u}}) = 0 \tag{3.55}$$

$$C_2 = u^2{}_{12} + u^2{}_{13} + u^2{}_{23} - u_{11}u_{22} - u_{11}u_{33} - u_{22}u_{33} \tag{3.56}$$

This equation does not depend on the coordinate system chosen, and thus:

$Tr(\bar{\bar{u}}) = u_{ii}, C_2$, and $\det(\bar{\bar{u}})$ are invariants.

There are two independent invariants of the second order:

$$[Tr(\bar{\bar{u}})]^2,\ C_2$$

Any linear combination of these invariants is also invariant, for example:

$$\sum_{ik} u_{ik}^2 = [Tr(\bar{\bar{u}})]^2 + 2C_2$$

The second-order expansion of F, which is a scalar, must be expressed as a function of the two quadratic independent invariants:

$$F = F_0 + \frac{\lambda}{2}[Tr(\bar{\bar{u}})]^2 + \mu \sum_{ik} u^2{}_{ik} \tag{3.57}$$

where λ and μ are called Lamé parameters. It is helpful to express the tensor as the sum of a diagonal matrix (uniform compression or expansion) and a matrix with a trace of zero (pure shear).

$$u_{ik} = \left(u_{ik} - \frac{1}{3}\delta_{ik}(Tr(\bar{\bar{u}})) \right) + \frac{1}{3}\delta_{ik}(Tr(\bar{\bar{u}})) \tag{3.58}$$

As $\sum_{ik}\left(u_{ik}^2\right)$ and $[Tr(\bar{\bar{u}})]^2$ are also invariants, it is easy to demonstrate (optional exercise) that:

$\sum_{ik}\left(u_{ik} - \frac{1}{3}\delta_{ik}(Tr(\bar{\bar{u}})) \right)^2$ is also invariant.

Using the fundamental relation (3.48), F can thus be written:

$$F = F_0 + \mu \sum_{ik}\left(u_{ik} - \frac{1}{3}\delta_{ik}(Tr(\bar{\bar{u}})) \right)^2 + \frac{K}{2}(Tr(\bar{\bar{u}}))^2 \tag{3.59}$$

$$K = \lambda + \frac{2}{3}\mu \tag{3.60}$$

With $\sigma_{ik} = \left(\frac{\partial F}{\partial u_{ik}} \right)_T$, we can directly find the relation between the stress and the strain tensors.

It is Hooke's law again, already expressed in simple form in (3.2):

$$\sigma_{ik} = 2\mu\left(u_{ik} - \frac{1}{3}Tr(\bar{\bar{u}})\delta_{ik} \right) + KTr(\bar{\bar{u}})\delta_{ik} \tag{3.61}$$

It is simple to calculate the trace of σ_{ik}:

$$Tr(\overline{\overline{\sigma_{ik}}}) = 3KTr(\bar{\bar{u}}) \tag{3.62}$$

Substituting this expression into (3.61) and for a body in uniform compression $\sigma_{ik} = -p\delta_{ik}$, we obtain the inverse relation:

$$u_{ik} = \frac{1}{9K}Tr(\bar{\bar{\sigma}})\delta_{ik} + \frac{1}{2\mu}\left(\sigma_{ik} - \frac{1}{3}Tr(\bar{\bar{\sigma}})\delta_{ik} \right) \tag{3.63}$$

Considering (3.63) and (3.22), we come to:

$$Tr(\bar{\bar{u}}) = \frac{\Delta V}{V} = -\frac{p}{K} \tag{3.64}$$

For small pressure and volume changes, this can be written:

$$\frac{1}{K} = -\left(\frac{\partial V}{\partial p} \right)\frac{1}{V} \tag{3.65}$$

$1/K$ represents the coefficient of uniform compression. In the same way, considering a pure shear, we can give a physical meaning to μ.

As a matter of fact, if $u_{12}=\theta/2$ as in Figure 3-5, the expression (3.61) yields:

$$\sigma_{12}=\mu\theta \text{ or } \frac{\sigma_{12}}{\mu}=\theta$$

where μ is the shear modulus. Finally, considering a homogeneous tensile stress $\sigma_{33}=\sigma$, it is readily shown that the expression (3.2) can be obtained (optional exercise):

$$u_{33}=\frac{\sigma}{E} \text{ with } E=\frac{9K\mu}{3K+\mu} \tag{3.66}$$

where $\boldsymbol{E}$ is the Young's modulus of elasticity. We equally obtain the transverse components of the deformation using equation (3.3):

$$u_{11}=u_{22}=-\nu\frac{\sigma}{E}$$

Then poisons ratio, ν, can be also be expressed in K and μ:

$$\nu=\frac{1}{2}\frac{3K-2\mu}{3K+\mu} \tag{3.67}$$

We can, then, introduce the following analysis. Since K and μ are always positive, ν can only vary between -1 ($K=0$ by (3.67)) and 1/2. From equations (3.66) and (3.67), we can derive equation (3.7) discussed in section 3.1:

$$K=\frac{E}{3(1-2\nu)}$$

The formulas (3.58) and (3.61) are usually written with the classic parameters E and ν:

$$\sigma_{ik}=\frac{E}{1+\nu}\left(u_{ik}+\frac{\nu}{1-2\nu}Tr(\bar{\bar{u}})\delta_{ik}\right) \tag{3.68}$$

$$u_{ik}=\frac{1}{E}\left((1+\nu)\sigma_{ik}-\nu Tr(\bar{\bar{\sigma}})\delta_{ik}\right) \tag{3.69}$$

Relations between elastic parameters in isotropic bodies

$$E=2\mu+\lambda(1-2\nu) \text{ and } -2\mu\nu+\lambda(1-2\nu)=0$$

lead to:

$$\nu=\frac{\lambda}{2(\lambda+\mu)} \quad \text{and} \quad E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \tag{3.70}$$

and inversely, we obtain equation (3.11) discussed in section 3.1.

$$\mu=\frac{E}{2(1+\nu)}$$

This also gives:

$$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)} \tag{3.71}$$

and:

$$\lambda+\mu=\frac{\mu}{1-2\nu} \tag{3.72}$$

3.6.2 Temperature effects

The preceding expressions have been introduced without considering thermal dilatation. However, if the temperature varies, deformation occurs without external stresses. Therefore, a term of the first order in $\boldsymbol{u_{ik}}$ has to be added to the expression (3.59) for the free energy $\boldsymbol{F}$. Moreover, we consider that the temperature variations are small, so the variation of F as a function of temperature is also of the first order. Therefore, the only first-order invariant being $\boldsymbol{Tr}\ (\bar{\bar{\boldsymbol{u}}})$, $\boldsymbol{F}$ must be in the form:

$$F = F_0(T) - K\alpha(T - T_0)Tr(\bar{\bar{u}}) + \mu\sum_{ik}\left(u_{ik} - \frac{1}{3}\delta_{ik}(Tr(\bar{\bar{u}}))\right)^2 + \frac{K}{2}(Tr(\bar{\bar{u}}))^2 \tag{3.73}$$

We obtain the expression of the stress tensor using equation (3.48) as before:

$$\sigma_{ik} = -K\alpha(T - T_0)\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}Tr(\bar{\bar{u}})\delta_{ik}\right) + KTr(\bar{\bar{u}})\delta_{ik} \tag{3.74}$$

If $\boldsymbol{\sigma_{ik}}$ is zero, considering the trace of the right-hand side of (3.74), we come to:

$$Tr(\bar{\bar{u}}) = \frac{\Delta V}{V} = \alpha(T - T_0) \tag{3.75}$$

where α is the coefficient of thermal expansion. The linearity of equation (3.74) is generally valid for a range of temperatures of some tens of degrees. The expression is valid for more significant variations, but we consider that $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{T})$.

3.6.3 Equilibrium equation of isotropic bodies

We can apply the equilibrium equation (3.26) to the expression for the stress tensor given by Hooke's law in (3.68):

$$\begin{aligned} \frac{\partial\sigma_{ik}}{\partial x_k} = 0 = \frac{E}{1+\nu}\left(\frac{\partial u_{ik}}{\partial x_k} + \frac{\nu}{1-2\nu}\frac{\partial Tr(\bar{\bar{u}})}{\partial x_i}\right) = \\ = \frac{E}{2(1+\nu)}\frac{\partial^2 u_i}{\partial x_k^2} + \frac{E}{2(1+\nu)(1-2\nu)}\frac{\partial^2 u_l}{\partial x_i \partial x_l} \end{aligned} \tag{3.76}$$

This equation can be written in the vectorial form:

$$(1-2\nu)\Delta\vec{u} + \overrightarrow{grad}(div(\vec{u})) = 0 \tag{3.77}$$

This expression can be found in the literature in various forms, depending on the vector calculus formulas used. For example, applying the relation, $\overrightarrow{grad}(div(\vec{u})) = \Delta\vec{u} + \overrightarrow{rot}(\overrightarrow{rot}(\vec{u}))$ equation (3.77) becomes:

$$2(1-\nu)\overrightarrow{grad}(div(\vec{u})) - (1-2\nu)\overrightarrow{rot}(\overrightarrow{rot}(\vec{u})) = 0 \tag{3.78}$$

Applying divergence to equation (3.78), we obtain:

$$\Delta div(\vec{u}) = 0 \tag{3.79}$$

Applying the Laplacian to equation (3.77) and taking into account (3.79):

$$\Delta\Delta\vec{u} \tag{3.80}$$

These expressions enable us to solve equilibrium problems for elastic bodies.

3.6.4 Generalized Hooke's Law

As both $\bar{\bar{\sigma}}$ and $\bar{\bar{u}}$ are tensors of order 2, Hooke's law is expressed for the general case by the linear relation:

$$\sigma_{ij} = C_{ijkl}u_{kl} \tag{3.81}$$

The tensor for the elastic constants $\bar{\bar{\bar{\bar{C}}}}$ is of order 4, i.e., it has $\mathbf{3^4 = 81}$ components. However, considering that the contribution to elastic energy:

$$W = \frac{1}{2}\sigma_{ij}u_{ij} = \frac{1}{2}C_{ijkl}u_{kl}u_{ij} \tag{3.82}$$

It must be equivalent if the term considered is either $\boldsymbol{u_{kl}u_{ij}}$ or $\boldsymbol{u_{ij}u_{kl}}$ implies that $\bar{\bar{\bar{\bar{C}}}}$ must be symmetric (with 45 different components). Furthermore, as $\bar{\bar{\sigma}}$ and $\bar{\bar{u}}$ are symmetric, we have:

$$\sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijlk} = C_{jilk}$$

and

$$u_{kl} = u_{lk} \Rightarrow C_{ijkl} = C_{jilk}$$

So, finally, there are only 21 independent components in the general case. The tensor $\bar{\bar{\bar{\bar{C}}}}$ can thus be reduced to a 6x6 representation in which it is usual to use the notations:

$$11 \rightarrow 1, 22 \rightarrow 2, 33 \rightarrow 3, 23 \rightarrow 4, 31 \rightarrow 5, 12 \rightarrow 6$$

However, to give a direct physical meaning to the constants $\boldsymbol{C_{ij}}$, the standard convention adopts from $u_4 = 2u_{23}, u_5 = 2u_{31}, u_4 = 2u_{12}$, which derives the representation that is called the Voigt notation:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

Finally, as the elastic constants result from the solid's atomic nature, the symmetry of this structure also reduces the number of components. The thermal expansion coefficient (see (3.73)) also becomes a tensor of order 2. From symmetry considerations, the number of independent constants is relatively limited (Table III-1).

Application to cubic crystals

a) elastic constants tensor

Consider crystals with a cubic structure and take the axis parallel to the cubic cell; the elastic constants must be invariant under every symmetry operation of the cube. We can thus show (see, for example, Hirth & Lothe, p. 35):

$C_{iiii} = C_{11}$ and for $i \neq j$ $C_{iijj} = C_{12}, C_{ijij} = C_{44}, C_{iiij} = 0$

For the complete characterization of a cubic crystal, it is sufficient to have three elastic constants corresponding to the only three possible independent modes of deformation (dilatation, pure shear, and simple shear). Therefore, the elastic constants tensor can be represented by:

$$C_{mn} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

From which we can write these relations between $\bar{\bar{\sigma}}$ and $\bar{\bar{u}}$:

$$\begin{aligned} \sigma_{11} &= C_{11}u_{11} + C_{12}u_{22} + C_{12}u_{33} \\ \sigma_{22} &= C_{12}u_{11} + C_{11}u_{22} + C_{12}u_{33} \\ \sigma_{33} &= C_{12}u_{11} + C_{12}u_{22} + C_{11}u_{33} \\ \sigma_{23} &= 2C_{44}u_{23} \\ \sigma_{31} &= 2C_{44}u_{31} \\ \sigma_{12} &= 2C_{44}u_{12} \end{aligned}$$

Table III-1: Minimal number of independent components of the elasticity modules tensor C_{ij} *and of* α_{ij} *the thermal expansion coefficients tensor*

Crystal system	C_{ij}	α_{ij}
triclinic	18	3
monoclinic	12	3
orthorhombic	9	3
tetragonal	6	2
rhombohedral	6	2
hexagonal	5	2
cubic	3	1

b) Anisotropy coefficient

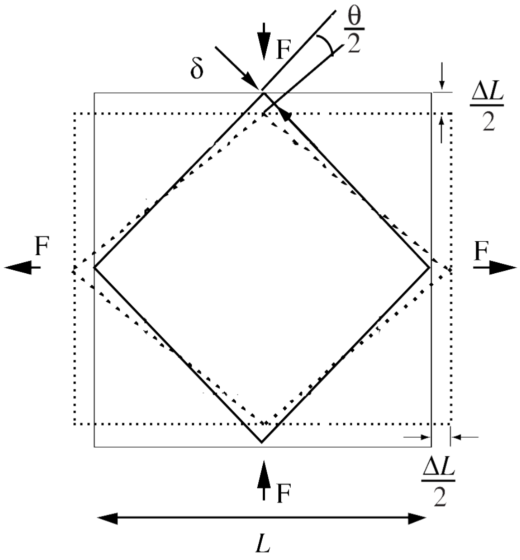


Figure 3-3: *Shear*

We have seen that in the pure shear case, we had in the coordinate system Ox_1x_2 (Figure 3-3):

$$\bar{\bar{\sigma}} = \begin{bmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

That implies the deformations:

$$\bar{\bar{u}} = \begin{bmatrix} 0 & u_{12} & 0 \\ u_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly to what we did for stresses, we can show that (in the isotropic case) the strain tensor can be diagonalized and that it has as its principal axis the coordinate system $Ox_1'x_2'x_3'$ in which it is written in the form:

$$\bar{\bar{u}}' = \begin{bmatrix} u_{11}' & 0 & 0 \\ 0 & u_{22}' & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $u_{11}' = u_{12}\ \ u_{22}' = -u_{12}$, which is very similar to the expression for stresses:

$$\bar{\bar{\sigma}}' = \begin{bmatrix} \sigma_{11}' & 0 & 0 \\ 0 & \sigma_{22}' & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $\sigma_{11}' = \sigma_{12}\ \ \sigma_{22}' = -\sigma_{12}$

From Hooke's law for a cubic crystal:

$$\sigma_{12} = 2C_{44}u_{12}$$
$$\sigma_{11}' = C_{11}u_{11}' - C_{12}u_{22}' = (C_{11} - C_{12})u_{12}$$

We also have $\sigma_{12} = \sigma_{11}'$, and in this case, the two elastic constants remain independent since:

$$2C_{44} = C_{11} - C_{12} \tag{3.83}$$

and (see eq. 3.61) $C_{44} = \mu$. We define then the *"Anisotropy factor" A* so that:

$$A = \frac{2C_{44}}{C_{11} - C_{12}} \tag{3.84}$$

For $A = 1$, the crystal is elastically isotropic. As shown in the table below, most crystals - even cubic ones - are relatively anisotropic.

Table III-2: *Anisotropy factors for different materials*

	Na	**K**	**Fe**	**W**	**Al**	**Cu**	**Pb**	**Cdiam**	**NaCl**	**KCl**
A	7.5	5.7	2.4	1	1.2	3.2	4	1.6	0.7	0.36

We also note that the so-called compliance tensor. $\boldsymbol{S_{ijkl}}$, relates strain and stress by:

$$u_{ij} = S_{ijkl}\sigma_{kl} \tag{3.85}$$

With the reduced index notation introduced before:

$S_{ijkl} = S_{mn}$ if m, n = 1, 2, 3

$S_{ijkl} = \frac{1}{2}S_{mn}$ if m ≠ n = 4, 5, 6

$S_{ijkl} = \frac{1}{4}S_{mn}$ if m = n = 4, 5, 6

We can then use classical matrices to change the basis for the matrix. Equation (3.85) becomes, for example, with reduced indices:

$$u_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 + \frac{1}{2}(S_{14}\sigma_4 + S_{15}\sigma_5 + S_{16}\sigma_6) \tag{3.86}$$